Novel Scalar-Vector Potential Formulation for Three-Dimensional, Inviscid, Rotational Flow Problems

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A computational method for the calculation of steady, strongly rotational, inviscid, subsonic flowfields in three-dimensional ducts is presented. The method is based on the decomposition of the velocity vector into a potential and rotational part through the Helmholtz theorem. The computational algorithm requires the solution of elliptic-type equations for the scalar and vector potentials, where a completely novel approach for solving the vector potential equation has been adopted. The new formulation has better physical meaning than our previous one and has the great advantage of decoupling the vector potential boundary conditions. The use of very fast elliptic solvers, based on preconditioned minimization techniques, leads to very economic computations. The transport equations are handled in their Lagrangian form.

Nomenclature

C_p = specific heat at constant pressure
C_v = specific heat at constant volume
$\boldsymbol{g}_i$ = covariant vector base
$\boldsymbol{g}^i$ = contravariant vector base
g^{ij} = contravariant metrics
$\boldsymbol{n}$ = outward unit normal vector
p = pressure
R_g = universal gas constant
S = entropy
T = temperature
γ = isentropic exponent
ξ,η,ζ = coordinates in computational space
ρ = density
Φ = scalar potential
$\boldsymbol{\Psi}$ = vector potential
$\Psi(i)$ = Cartesian $\boldsymbol{\Psi}$ components
Ψ_i = covariant $\boldsymbol{\Psi}$ components
$\boldsymbol{\Omega}$ = vorticity vector

Subscripts

t = total thermodynamic quantities
n = normal component
s = streamwise component

I. Introduction

DURING the last decade, several computational methods have been developed for the solution of inviscid flow problems in turbomachines, blade passages, and air inlets. When rotational flow effects are present, the full potential approximation seems inadequate and two families of approaches may be established. The first contains methods based on the direct solution of the primitive-variable equations, whereas the second uses convenient transformations that lead to potential-type formulations.

In primitive-variable formulations,[1] the time-dependent terms in the governing equations are retained even for the steady-state case and the system of equations is purely hyperbolic. When an explicit scheme is applied, the Courant–Friedrichs–Lewy criterion sets some limitations for the allowed computational time step and thus increases the computational cost. For low subsonic flows, convergence becomes slower, and in some cases, these methods are rather uneconomical. The convergence characteristics of a primitive-variable solver certainly may be improved when a fully implicit method is used. In that case, however, the complexity of the numerical scheme and computer memory requirements increase sufficiently. A second drawback of the primitive-variable solvers is the relatively poor estimation of the total enthalpy and entropy fields because of the diffusing effects of the artificial viscosity terms needed for the convergence.

The appearance of fast and accurate algorithms, capable of solving elliptic-type differential equations, gave rise to the development of potential-type formulations, depending on the number of potential functions used to express the velocity vector, as well as the kind of additional unknowns. Three subclasses of methods can be defined.

1) Methods that use the Clebsch transformation or its particular forms developed by Casal.[2] The decomposition of the velocity vector in terms of a scalar and vector potential was used by Lacor and Hirsch[3] for compressible stator flows and incompressible rotor ones and by Ecer et al.[4] for isoenergetic flows. In both references, the equations to be solved are cast in elliptic forms.

2) Methods that employ the decomposition of the three-dimensional flowfield into three sets of orthogonal nonstream surfaces. On each surface, two-dimensional flow problems are solved simultaneously using three streamlike functions.[5,6]

3) Methods based on the Helmholtz theorem,[7,8] according to which the velocity vector is decomposed in terms of the gradient of a scalar potential and the curl of a vector potential.

A method for the calculation of three-dimensional, steady, rotational, inviscid flow problems in ducts, which was based on the Helmholtz theorem, has been presented by the authors[9] in the past. An extension of that method is now presented, which allows a more convenient robust formulation of the vector potential equations. It is the velocity vector that is decomposed instead of the flux vector, and the elliptic-type equations are solved in terms of the covariant components of the vector potential. With this modification, the problem of coupled boundary conditions, when dealing with a stream function solver, has been simplified and a set of uncoupled boundary conditions derived, which easily can be incorporated implicitly in the discretization scheme. Thus, the stability of the iterative solver has been improved and faster convergence rates achieved.

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The generalized minimal residual (GMRES) method, combined with appropriate preconditioning techniques, is used to solve numerically the elliptic-type equations. The transport equations are handled in their Lagrangian form.

II. Governing Equations

Our analysis is based on the decomposition of the velocity vector field V through Helmholtz's theorem into an irrotational and solenoidal part[10]

$$V = \nabla\Phi + \nabla \times \Psi \tag{1}$$

Applying the $\nabla \cdot \rho(\,)$ and $\nabla \times (\,)$ operators on Eq. (1), considering the continuity equation $\nabla \cdot (\rho V) = 0$ and vorticity definition $\Omega = \nabla \times V$, and letting Ψ satisfy the relation

$$\nabla \cdot \Psi = 0 \tag{2}$$

which is a necessary and sufficient condition for the uniqueness of the decomposition,[9] we obtain

$$\nabla \cdot (\rho \nabla\Phi) + \nabla \cdot (\rho \nabla \times \Psi) = 0 \tag{3}$$

$$-\nabla^2 \Psi = \Omega \tag{4}$$

The vector product of the Lamb equation and velocity vector V yield[11]

$$\Omega = \rho V I + F \tag{5}$$

$$I = \frac{\Omega_s}{\rho V} \tag{6}$$

where Ω_s is the streamwise vorticity component (called secondary vorticity) and F is the normal to the velocity vector vorticity component defined as

$$F = \left(\frac{1}{V^2}\right)[V \times (T\nabla S - C_p \nabla T_t)] \tag{7}$$

Using the continuity equation, in combination with the divergence-free property of the vorticity field, one easily may see from Eq. (5) that

$$V \cdot \nabla I = -\left(\frac{1}{\rho}\right)\nabla \cdot F \tag{8}$$

For a steady inviscid, adiabatic, and shockless flow, the energy and entropy conservation laws are expressed in the following form:

$$V \cdot \nabla T_t = 0 \tag{9}$$

$$V \cdot \nabla S = 0 \tag{10}$$

declaring the stagnation temperature and entropy conservation along the flowfield streamlines.

The foregoing system of equations is closed by the Bernoulli equation

$$\left(\frac{\rho}{\rho t}\right)^{\gamma - 1} + \frac{V^2}{2C_p T_t} = 1 \tag{11}$$

and the equation of state for a perfect gas

$$S - S_{\text{ref}} = C_v \ell n\left[\left(\frac{\rho_{t\text{ref}}}{\rho_t}\right)^{\gamma - 1}\left(\frac{T_t}{T_{t\text{ref}}}\right)\right] \tag{12}$$

where ref indicates a reference condition.

III. Boundary Conditions

The system of equations (1–12) consists of four elliptic-type partial differential equations (one for the scalar potential Φ and one for each component of the vector potential Ψ) and three transport-type equations for the streamwise vorticity component I and the thermodynamic quantities T_t and S.

To solve the transport equations, the I, T_t, and S distributions at the entry part of the flowfield must be specified. According to Hirasaki et al.[12] and Richardson et al.,[13] we may impose as a boundary condition on Φ

$$n \cdot \nabla\Phi = \frac{\partial\Phi}{\partial n} = V \cdot n = V_n \tag{13}$$

where n is the outward, normal to the boundary, unit vector. Combining Eqs. (1) and (13), we form the following boundary condition on Ψ:

$$n \cdot \nabla \times \Psi = 0$$

which reduces to

$$n \times \Psi = 0 \tag{14}$$

Equation (14) states that the tangential component of Ψ on the boundary surface has to be zero. The assumption that Ψ is a divergence-free vector field is also applied in the near-boundary region, thus providing the boundary condition for the normal Ψ component.

Equation (13) requires the knowledge of the normal velocity V_n component distribution over the whole boundary surface of the computational domain. Since the final set of equations that needs to be solved consists of transport- and elliptic-type partial differential equations, it has to be pointed out that irregularities at the exit V_n distribution are not propagated at the interior of the domain and thus influence only the grid lines adjacent to the exit. It is evident that when the boundary surface coincides with an impermeable wall, boundary condition (13) simplifies to

$$\frac{\partial\Phi}{\partial n} = 0$$

IV. Governing Equations in Transformed Coordinates

The governing equations are transformed from the arbitrary domain in physical space (x_i; $i = 1, 2, 3$) to the Cartesian-like domain in computational space (u_i; $i = 1, 2, 3$) through the general independent variable transformation

$$u_i = u_i(x_j); \qquad i, j = 1, 2, 3 \tag{15}$$

A. Elliptic-Type Equation for Φ

The scalar potential equation, written in the computational domain, is given by

$$\frac{1}{J}\frac{\partial}{\partial u_i}(J\rho U^i) = -\frac{1}{J}\frac{\partial}{\partial u_i}(J\rho \Lambda^i) \tag{16}$$

where U^i and Λ^i are the contravariant components of the scalar and vector potential parts, respectively,

$$U^i = (\nabla\Phi)^i = g^{ij}\frac{\partial\Phi}{\partial u_j}$$
$$\Lambda^i = (\nabla \times \Psi)^i \tag{17}$$

and g^{ij} is the contravariant tensor.

The Neumann-type boundary conditions on Φ are expressed on any u_k = constant surface as

$$u^k = V_n\sqrt{g^{kk}} \tag{18}$$

B. Poisson-Type Equation for Ψ

Instead of solving Eq. (4) in terms of the Cartesian Ψ components, as presented in our previous approach,[9] a new formulation is attempted here. According to this, Eq. (4) is first cast into a new form in terms of the covariant Ψ components and then solved with a simplified set of boundary conditions.

Applying the Laplace operator on the relation

$$\Psi_i = \frac{\partial x_j}{\partial u_i}\Psi(j); \qquad i,j = 1, 2, 3$$

where Ψ_i and $\Psi(i)$ are the covariant and Cartesian Ψ components, respectively, we get

$$\nabla^2(\Psi_i) = -\Omega_i + 2\nabla[\Psi(j)]\cdot\nabla\left(\frac{\partial x_j}{\partial u_i}\right) + \Psi(j)\nabla^2\left(\frac{\partial x_j}{\partial u_i}\right)$$

$$= -\Omega_i + 2\frac{\partial}{\partial u_\mu}[\Psi(j)]\frac{\partial^2 x_j}{\partial u_\nu u_i}g^{\mu\nu} + \Psi(j)\nabla^2\left(\frac{\partial x_j}{\partial u_i}\right)$$

$$= -\Omega_i + \frac{\partial \Psi_j}{\partial u_k}\alpha^i_{jk} + \Psi_k\beta^i_j \tag{19}$$

where α^i_{jk} and β^i_j are simple expressions of the second kind of Christoffel symbols

$$\alpha^i_{\lambda\mu} = \begin{Bmatrix}\lambda\\ vi\end{Bmatrix}g^{\mu\nu}$$

$$\beta^i_\lambda = \frac{\partial}{\partial u_\mu}\begin{Bmatrix}\lambda\\ vi\end{Bmatrix}g^{\mu\nu} - \begin{Bmatrix}\lambda\\ \mu p\end{Bmatrix}\begin{Bmatrix}p\\ vi\end{Bmatrix}g^{\mu\nu} - \begin{Bmatrix}p\\ \mu\nu\end{Bmatrix}\begin{Bmatrix}\lambda\\ pi\end{Bmatrix}g^{\mu\nu} \tag{20}$$

The left-hand-side member of Eq. (19), representing the Laplacian operation on (Ψ_i), takes the form

$$\nabla^2(\Psi_i) = \frac{1}{J}\frac{\partial}{\partial u_j}\left[Jg^{jk}\frac{\partial}{\partial u_k}(\Psi_i)\right] = \frac{1}{J}\frac{\partial}{\partial u_j}(JU^j_i) \tag{21}$$

where the contravariant components U^j_i are defined as

$$U^j_i = g^{jk}\frac{\partial}{\partial u_k}(\Psi_i) \tag{22}$$

We may observe that, in general, the right-hand side of Eq. (19) is a mixed linear function of the $\Psi(j)$; $j = 1, 2, 3$ components. In the very special case of a linear coordinate transformation, in which the Christoffel symbols are identically zero, Eqs. (19) can be solved independently.

To express the boundary conditions on Ψ, we proceed as follows: Let $u_k =$ constant be a boundary surface. Then, Eq. (14) yields

$$\frac{\Psi_i g^i \times g^k}{\sqrt{g^{kk}}} = 0 <=> \Psi_i(g^i \times g^k) = 0 \quad \text{or} \quad \Psi_i = 0, \quad \forall i \neq k \tag{23}$$

The boundary condition on the normal Ψ_k component results from the divergence-free condition on Ψ.

$$f_k\Psi_k + U^k_k = -g^{k\lambda}\frac{\partial}{\partial u_k}(\Psi_\lambda); \quad \lambda = k, \quad \text{no summation on } k \tag{24}$$

where $f_k = \nabla^2 u_k$.

A typical application of Eqs. (23) and (24) could be the $u_1 =$ constant surface; for that surface, the two tangential Ψ components have to be zero

$$\Psi_2 = \Psi_3 = 0$$

whereas for the Ψ_1 component, a Robin-type boundary condition will be imposed, which may be written in the form

$$\alpha\Psi_1 + \beta\left(\frac{\partial \Psi_1}{\partial \eta}\right) = \gamma$$

The three coefficients α, β, and γ are those derived through the development of Eq. (24).

Note that for a boundary orthogonal coordinate transformation, the boundary condition on the normal Ψ_k component does not depend on the "tangential" Ψ_λ components. If, in addition, u_k is a harmonic function, boundary condition (24) reduces to a simple Neumann-type boundary condition on Ψ_k.

C. Transport Equations on T_t, S, and I

The transport equations will be handled in their Lagrangian form. The substantial derivative $D(\ )/dt$ that measures the rate of change of $(\ )$ along the flowfield trajectories (streamlines for the steady-state case) is expressed as

$$\frac{D(\ )}{dt} = \frac{\partial(\ )}{\partial t} + V\cdot\nabla(\ ) = V\cdot\nabla(\ )$$

In the transformed space u_i, the substantial derivation operand is written as

$$\frac{D}{dt}(\ ) = V\cdot\nabla(\ ) = V^i g_i\cdot g^j\frac{\partial}{\partial u_j}(\ ) = V^i\frac{\partial}{\partial u_j}(\ )\delta^j_i$$

$$= V^i\frac{\partial(\ )}{\partial u_i} = \frac{D^*(\ )}{dt} \tag{25}$$

$$V^i = \frac{1}{J}\left[\frac{\partial}{\partial u_i}\left(\rho J g^{i\lambda}\frac{\partial\Phi}{\partial u_\lambda}\right) + \varepsilon^{ijk}\left(\frac{\partial\Psi_k}{\partial u_j} - \frac{\partial\Psi_j}{\partial u_k}\right)\right] \tag{26}$$

where V^i is the velocity vector contravariant component, δ^j_i the Kronecker delta, and ε^{ijk} the permutation symbol. Operator $D^*(\ )/dt$ measures the rate of change along the streamlines of the transformed flowfield.

If we consider that the velocity field is known, the T_t, S, and I distributions prove to be the solution of the following system of ordinary differential equations:

$$\frac{D^*u_i(t)}{dt} = V^i[u_j(t)]; \qquad i,j = 1, 2, 3$$

$$\frac{D^*T_t(t)}{dt} = 0$$

$$\frac{D^*S(t)}{dt} = 0$$

$$\frac{D^*I(t)}{dt} = c_2[u_j(t)] \tag{27}$$

with the initial conditions $(u_i)_0$, $(T_t)_0$, $(S)_0$, and $(I)_0$ for $t = 0$.

V. Solution Algorithm

We propose the following "fixed-point algorithm" to solve the nonlinear system of Eqs. (1–12).

Initialization

A first estimation of the flowfield is obtained through the solution of the full potential equation

$$\nabla\cdot(\rho\nabla\Phi) = 0, \qquad \rho = \rho(\Phi)$$

with the Neumann-type boundary conditions (13).

External Iterations

Step 1

Perform the necessary number of "internal iterations" to calculate the Φ field through Eq. (3). This step is obviously

omitted during the first external iteration, where the initialization is activated instead.

Step 2

Estimate the velocity field through the current Φ and Ψ values. Calculate the corresponding streamlines and transport the thermodynamic quantities T_t and S and the I vorticity component along them.

Step 3

Calculate the vorticity vector field Ω using Eq. (5).

Step 4

Determine the rotational flow part by solving (internal iterations) Eqs. (19) for the three Ψ covariant components and update the velocity field.

Step 5

Compute the local values of the stagnation density through Eq. (12) and update the static density through Eq. (11).

Step 6

Repeat steps 1 to 5 until a satisfactory convergence is obtained.

The foregoing iterative scheme has proven to be reliable in the subsonic case, in which the linearization of the density and vorticity effects seems to be a reasonable assumption.

VI. Discretization and Numerical Solution of the Partial Differential Equations

We consider a discretization of the transformed three-dimensional region, such as

$$u_1 = i\Delta u_1; \quad 1 \le i \le IT$$

$$u_2 = j\Delta u_2; \quad 1 \le j \le JT; \quad \Delta u_1 = \Delta u_2 = \Delta u_3 = 1; \quad i, j, k \text{ integers}$$

$$u_3 = k\Delta u_3; \quad 1 \le k \le KT$$

If we assume a linear distribution of the (JU_μ^λ) quantities into the cubic grid cells, which corresponds to a quadratic distribution of Φ and Ψ in them, the discretization of the potential-type operator gives

$$L_0(\rho^\lambda) = \frac{1}{J}[(\rho^\lambda JU^1)_{i+1/2,j,k} - (\rho^\lambda JU^1)_{i-1/2,j,k} + (\rho^\lambda JU^2)_{i,j+1/2,k} - (\rho^\lambda JU^2)_{i,j-1/2,k} + (\rho^\lambda JU^3)_{i,j,k+1/2} - (\rho^\lambda JU^3)_{i,j,k-1/2}]$$

$(\rho^\lambda JU^i)$ are discretized by central finite-difference schemes, and λ is the density exponent such that $\lambda = 0$ for the Ψ_i equations and $\lambda = 1$ when solving in terms of the scalar potential Φ.

The metrics g^{ij} and the Jacobian J are discretized on the midnodes with second-order accurate finite-difference schemes. According to the foregoing analysis, the discretization of the scalar potential equation with Neumann boundary conditions on Φ forms a linearized algebraic system that in the most general form can be written as

$$[L(\rho)]\{\Phi\} = \{f_p\} \tag{28}$$

The vector $\{f_p\}$ expresses the influence of the source terms, due to the incoming and outcoming mass flux. The term $[L(\rho)]$ is the full potential matrix in which the Neumann-type boundary conditions are incorporated implicitly.[14] This is a square, nonsymmetric matrix that has its nonzero elements on 19 symmetrically ordered diagonals.

Equation (19) when discretized forms three Laplacian matrices $[L_1]$, $[L_2]$, and $[L_3]$ corresponding to the left-hand-side member of Eq. (19) for the Ψ_1, Ψ_2, and Ψ_3 unknowns, respectively. As previously shown the $[L_i]$ matrices contain implicitly the Ψ_i boundary conditions. The Laplacian operator is common to the three equations, which means that

$$[L_1] = [L_2] = [L_3]$$

for all the internal nodes. For each boundary node, two Dirichlet-type and one Neumann-type boundary conditions are assigned. Only the Neumann condition requires the existence of a discretized operator, and considerable economy in storage requirements is achieved. This is due to the fact that only a single $[L_\psi]$ matrix has to be stored, whereas the $[L_i]$; $i = 1, 2, 3$ matrices are easily obtained through the following relations:

$$[L_1] = [L_2] = [L_3] = [L_\psi] \quad \text{for the internal nodes}$$

$$[L_1] = [L_\psi] \quad \text{for the } i = 1 \text{ or } i = IT \text{ boundaries}$$

$$[L_2] = [L_\psi] \quad \text{for the } j = 1 \text{ or } j = JT$$

$$[L_3] = [L_\psi] \quad \text{for the } k = 1 \text{ or } k = KT$$

The derivatives $\partial\Psi_j/\partial u_k$ allocated at the right-hand side of (19) are discretized by second-order accurate, centered finite-difference schemes. Here, we form a matrix, called G, consisting of 63 diagonals, symmetrically ordered with respect to the main diagonal, whose elements are simple functions of the $\alpha^i_{\lambda\mu}$ and β^i_λ coefficients.

Equation (19) may now be written in the following matrix form:

$$\begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix} - \begin{bmatrix} & G & \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix} = -\begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} \tag{29}$$

This linear system is solved by an iterative scheme, based on the preconditioned linear GMRES procedure.[15,16] Considering Eqs. (29) in their residual form

$$R_i(\Psi_1,\Psi_2,\Psi_3) = L_i(\Psi_i) - G_i(\Psi_1,\Psi_2,\Psi_3) + \Omega_i = 0; \quad i = 1, 2, 3 \tag{30}$$

and assuming the GMRES(m) technique, we solve the following equivalent system:

$$N_i^{-1} R_i(\Psi_1,\Psi_2,\Psi_3) = 0; \qquad i = 1, 2, 3 \tag{31}$$

where N_i, $i = 1, 2, 3$ are preconditioning matrices. When approximate-factorization (ADI)[17] approximations of the $[L_i]$ matrices are used as preconditioners, no extra core memory in the computer is needed. We shall call this preconditioned GMRES solver a GMRES(m)/ADI scheme.[15]

The Ψ solver just presented has proven to be very efficient and also very economical in terms of computer memory requirements {19 arrays ($ITXJTXKT$) for the $[L_\psi]$ matrix and 36 arrays ($ITXJTXKT$) for the $\alpha^i_{\lambda\mu}$ and β^i_λ coefficients that form the G matrix}.

System (28) is solved in its equivalent preconditioned form

$$([L^{-1}][L(\rho)][U^{-1}])([U]\{\Phi\}) = [L^{-1}]\{f_p\} \tag{32}$$

by a linear GMRES(m) scheme. Matrices $[L]$ and $[U]$ are the same lower and upper diagonal matrices that one forms for the MSIP approximate-factorization procedure.[18,19] We call this preconditioned GMRES solver a GMRES(m)/MSIP scheme. Note that for the low subsonic case, matrices $[L]$ and $[U]$ may be computed by the approximate factorization of the Laplacian matrix $[L]$ instead of the full potential one $[L(\rho)]$.

The solution of the transport equations has been presented in Ref. 9.

VII. Results and Discussion

The method described here can be applied to the calculation of steady, compressible, rotational flows in three-dimensional duct geometries. The method's accuracy has been demonstrated already in Ref. 9, where comparative results were presented for the accelerating elbow of Stanitz. In comparison with Ref. 9, the new elements added here are the following:

1) The analysis of the velocity vector field instead of the mass flux vector field. This fact ameliorates the behavior of the code in the high subsonic flow regime, as it avoids the split of the ρV part in the static density calculation, thus maintaining the strongly conservative form of the continuity equation.

2) The Ψ solver is completely upgraded by covariant components analysis and the implicit treatment of its boundary conditions. This fact is crucial when we perform calculations on very stiff grids, where it was noticed that any explicit treatment of the boundary conditions was catastrophic for the convergence attitude of the Ψ solver.

3) All the elliptic solvers have been upgraded[15] by the use of preconditioned GMRES techniques. We, thus, obtain deeper convergence in less computer time, in comparison with the classical ADI and MSIP schemes. In addition, the GMRES scheme may handle better the coupled form of the vector potential equations, which are now solved simultaneously, without extra core memory requirements.

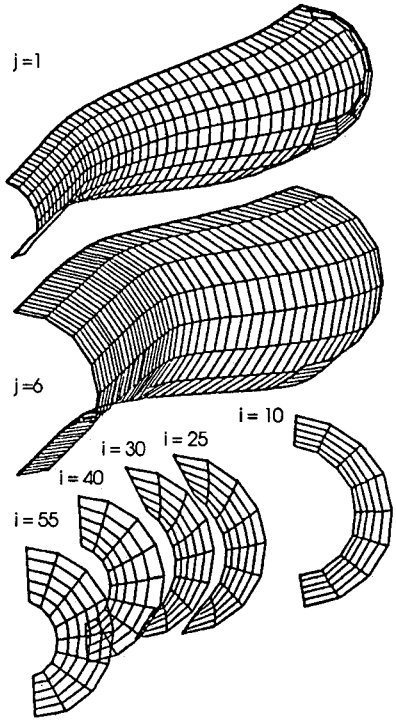

Fig. 1 Three-dimensional grid. Projections of different mesh sections.

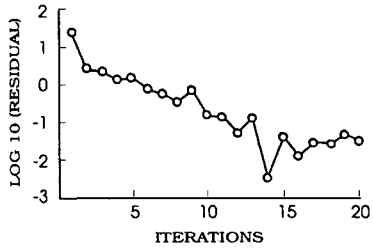

Fig. 2 Modified full potential solver. Convergence history [GMRES (5)/MSIP].

The current application concerns the flowfield calculation in a very complex, fully three-dimensional duct geometry, where the complete description of boundary conditions has already been determined through a primitive-variable code. The computational grid, having $55 \times 6 \times 9$ nodal points, is given in Fig. 1 and was constructed by a typical three-dimensional Laplacian grid generator. The same grid was used for the calculation of an irrotational, potential flow in Ref. 15, where the acceleration properties of the GMRES scheme were demonstrated. Here, however, the flow is compressible and rotational, since a nonuniform stagnation density distribution is imposed at the duct's inlet. The inlet stagnation temperature distribution was supposed to be uniform. The initialization of the flowfield is achieved by a full potential solution. The convergence history of that GMRES(5)/MSIP full potential solver is presented in Fig. 2. The convergence history of the complete algorithm is given in Fig. 3 in terms of the maximum residuals of the three vector potential equations, as functions of the external iteration number. We have to point out that three internal GMRES(5)/MSIP iterations for the Φ solver and three internal GMRES(3)/ADI iterations for the Ψ solver were performed during each external iteration. The overall computational cost on a MICROVAX/ULTRIX 32 machine was 12,295 CPU s. The calculated Mach number distribution on the duct's boundary surface is given in Fig. 4. The isostagnation density contours (coinciding with the isentropic contours) are presented in Fig. 5. In this figure, we notice the strongly convective character of the Lagrangian solver we use for the solution of the transport equation. A Eulerian solver of the transport equations may also be used instead,[20] but it is more time-consuming, as one has to solve

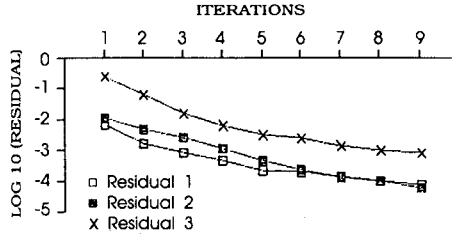

Fig. 3 Convergence history of the rotational solver. Residuals after each external iteration.

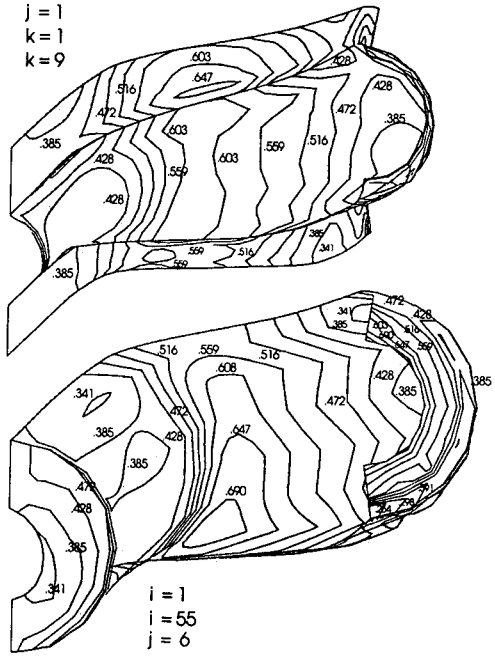

Fig. 4 Computed iso-Mach contours on the air inlet boundary.

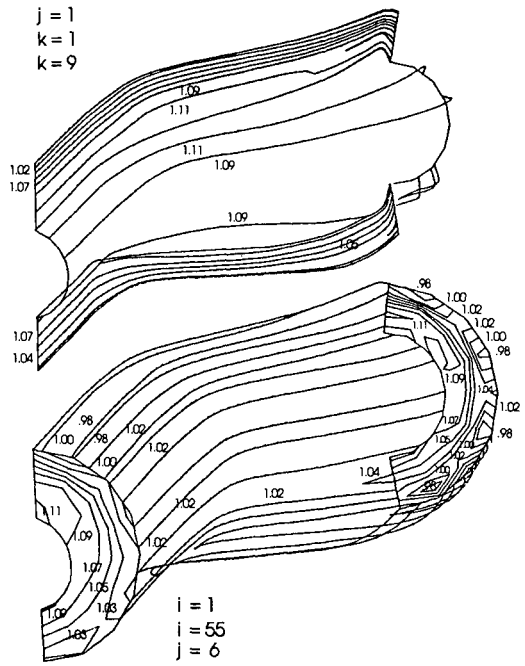

Fig. 5 Computed isostagnation density contours.

the equations independently. This is, in fact, one of the major advantages of this method, which permits a very accurate calculation of the entropy field. On the contrary, when a primitive-variable Euler solver is used, the diffusing character of the scheme influences the accuracy of the entropy and energy field calculation.

VIII. Conclusions

A method has been presented for solving the subsonic Euler equations in rectangular ducts. This method uses the Helmholz decomposition technique to split the velocity vector field into a potential and solenoidal part. The basic aspects of the computational scheme were presented in our previous work[9] where the methodology was successfully validated. In this article, we mainly attempt to upgrade our solver and to present an application for a realistic fully three-dimensional configuration. To accomplish this, we had to adopt a completely new approach for the vector potential equation, which is more "physical" than any Cartesian representation. Such is the covariant Ψ components formulation, which gives one a better understanding of the vector potential behavior on the transformed computational plane, uncouples the Ψ boundary conditions, and permits their implicit treatment in our finite-volume scheme. We have, moreover, proved that the Ψ partial differential equations are coupled through simple expressions of the second kind of Christoffel symbols that become known quantities after the coordinate transformation. This new approach increases the robustness of our method and its accuracy in the near-boundary region, and the adoption of fast elliptic solvers based on preconditioned minimization techniques improves impressively the convergence characteristics of the solution and, thus, diminishes computational cost. One of the main advantages of the method presented is the very accurate calculation of the energy and entropy fields due to the direct solution of the stagnation temperature and entropy conservation equations by a Lagrangian solver, which detects the flowfield streamlines. The present method is, for the time being, limited to subsonic flow calculations. Its extension to transonic applications, possibly through the use of an appropriate artificial density scheme, could be attempted in the near future.

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